Tutorial 3

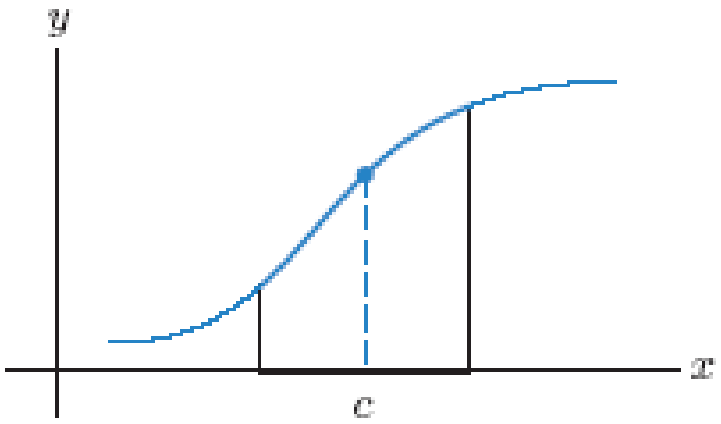
Applications of the Derivative

Outline

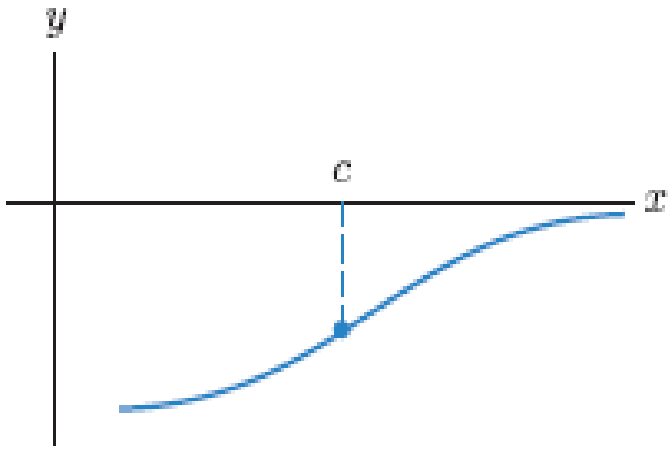
- Describing Graphs of Functions
- The First and Second Derivative Rules
- The First and Second Derivative Tests
- Optimization Problems

Increasing Functions

Increasing Functions We say that a function $f(x)$ is *increasing in the interval* if the graph continuously rises as x goes from from left to right through the interval. That is, whenever x_1 and x_2 are in the interval with $x_1 < x_2$, then we have $f(x_1) < f(x_2)$. We say that $f(x)$ is increasing at $x = c$ provided that $f(x)$ is increasing in some open interval on the x-axis that contains the point c.



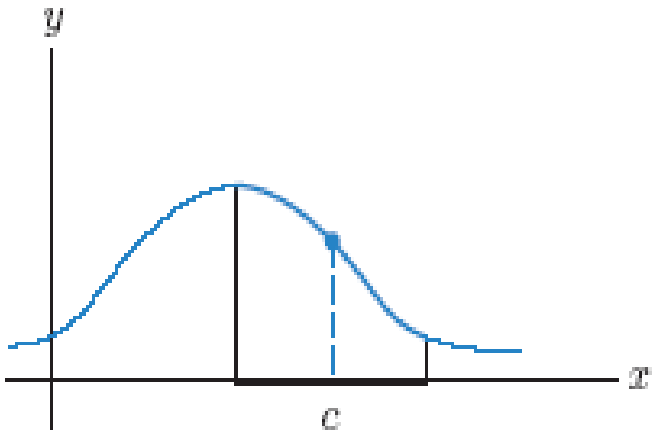
Increasing at $x = c$
(increasing in an
interval containing c)



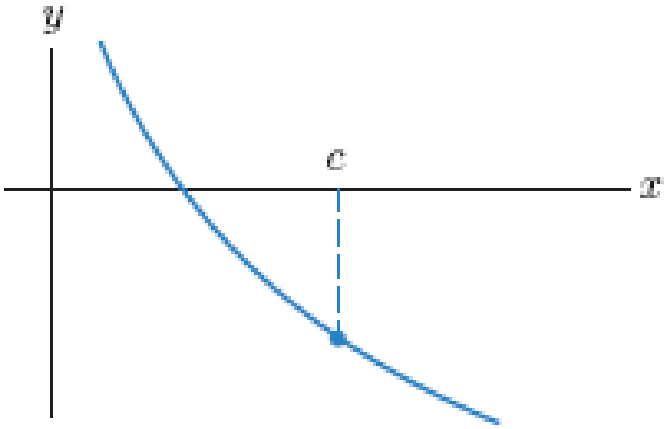
Increasing function at $x = c$
(values of $f(x)$
become less negative)

Decreasing Functions

Decreasing Functions We say that a function $f(x)$ is *decreasing in an interval* provided that the graph continuously falls as x goes from left to right through the interval. That is, whenever x_1 and x_2 are in the interval with $x_1 < x_2$, then we have $f(x_1) > f(x_2)$. We say that $f(x)$ is decreasing at $x = c$ provided that $f(x)$ is decreasing in some open interval that contains the point c.



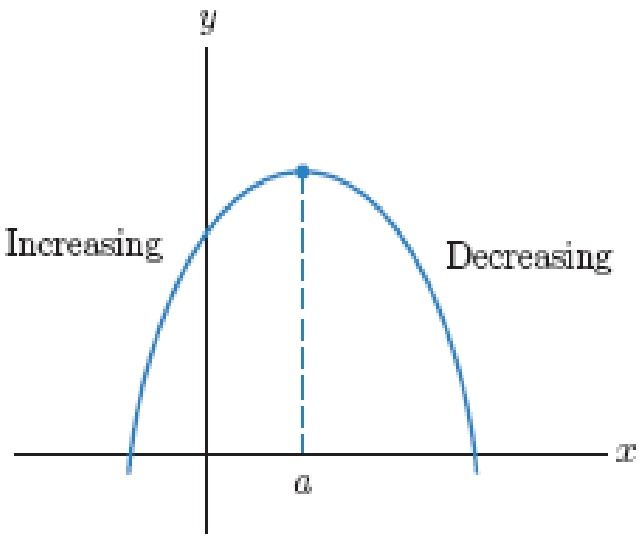
Decreasing at $x = c$
(decreasing in an
interval containing c)



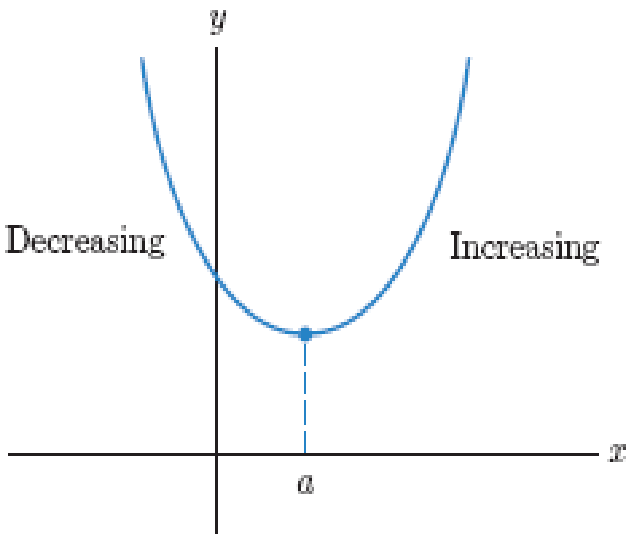
Decreasing function at $x = c$
(values of $f(x)$
become more negative)

Relative (Local) Maxima & Minima

Relative Maximum and Minimum A *relative maximum point* is a point at which the graph changes from increasing to decreasing; a *relative minimum point* is a point at which the graph changes from decreasing to increasing. (See Fig. 3.)



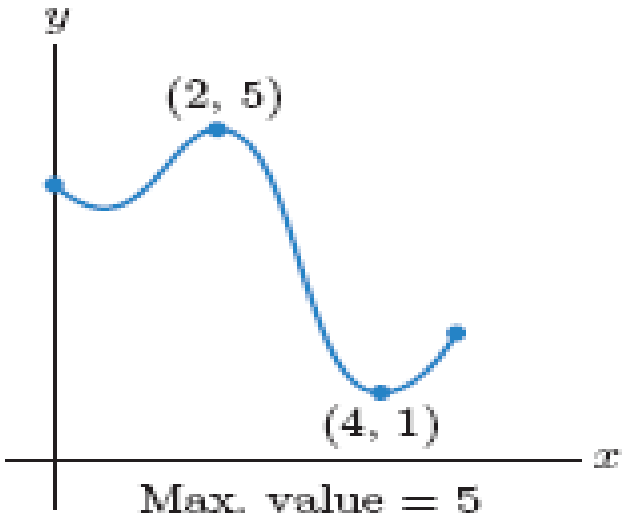
Relative maximum at $x = a$



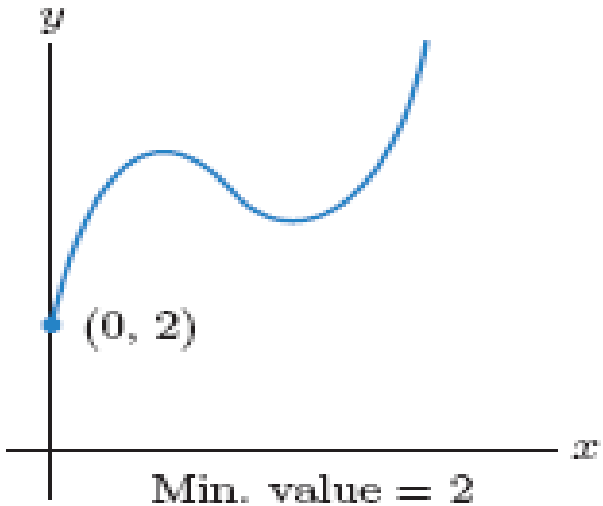
Relative minimum at $x = a$

Absolute (Global) Maxima & Minima

Absolute Maximum and Minimum The *maximum value* (or *absolute maximum value*) of a function is the largest value that the function assumes on its domain. The *minimum value* (or *absolute minimum value*) of a function is the smallest value that the function assumes on its domain.



Max. value = 5
Min. value = 1

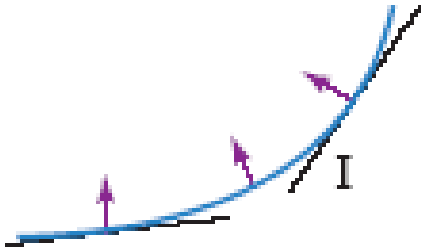


Min. value = 2
No max. value

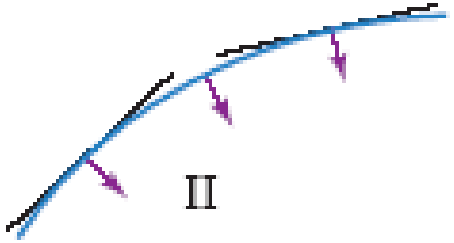
Concavity

Concavity We say that a function $f(x)$ is *concave up* at $x = a$ if there is an open interval on the x-axis containing a throughout which the graph of $f(x)$ lies above its tangent line. Equivalently, $f(x)$ is concave up at $x = a$ if the slope of the graph increases as we move from left to right through $(a, f(a))$.

Similarly, we say that a function $f(x)$ is *concave down* at $x = a$ if there is an open interval on the x-axis containing a throughout which the graph of $f(x)$ lies below its tangent line. Equivalently, $f(x)$ is concave down at $x = a$ if the slope of the graph decreases as we move from left to right through $(a, f(a))$. The graph in



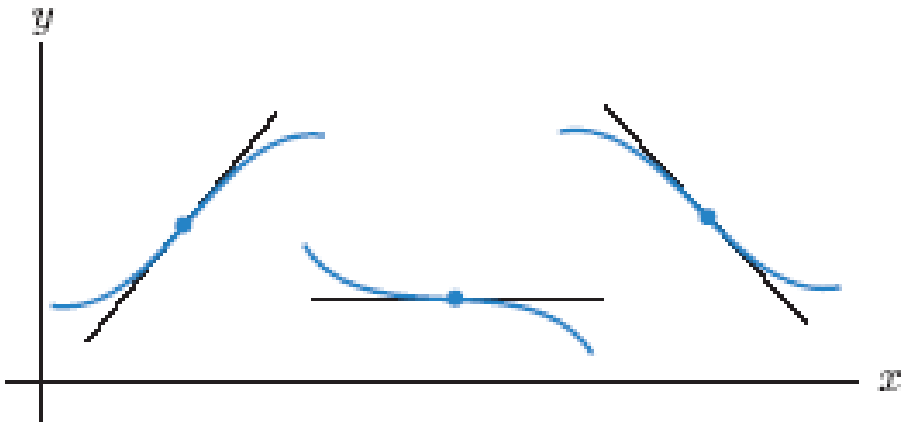
Concave Up



Concave Down

Inflection Points

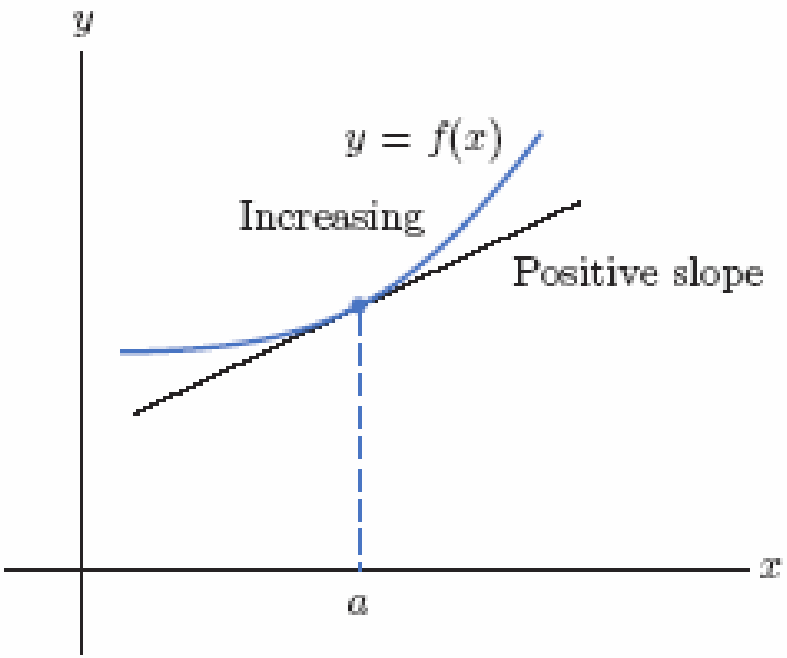
Inflection Point An *inflection point* is a point on the graph of a function at which the function is continuous and at which the graph changes from concave up to concave down, or vice versa. At such a point, the graph crosses its tangent line (Fig. 11). (The continuity condition means that the graph cannot break at an inflection point.)



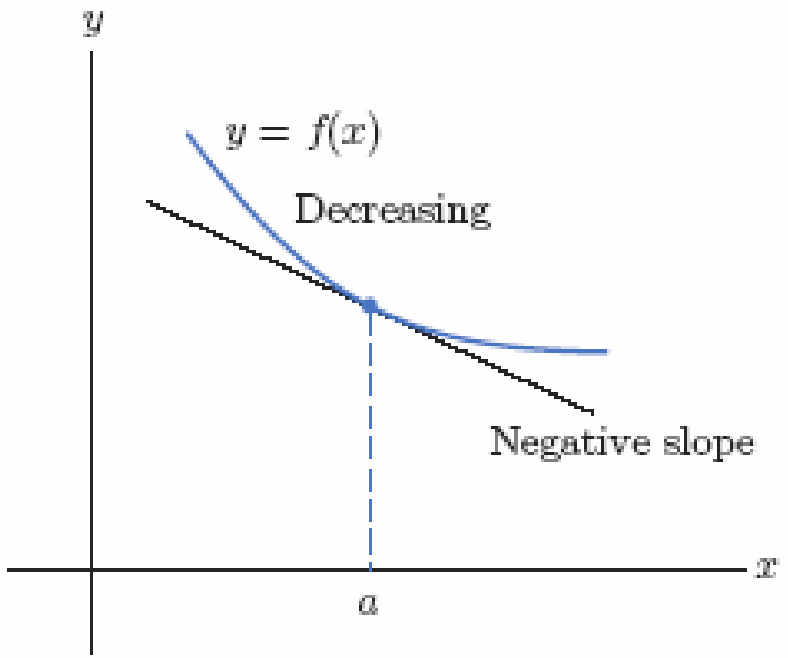
Notice that an inflection point is not where a graph changes from an increasing to a decreasing slope, but where the graph changes its concavity.

First Derivative Rule

First Derivative Rule If $f'(a) > 0$, then $f(x)$ is increasing at $x = a$. If $f'(a) < 0$, then $f(x)$ is decreasing at $x = a$.



$f'(a) > 0$



$f'(a) < 0$

First Derivative Rule

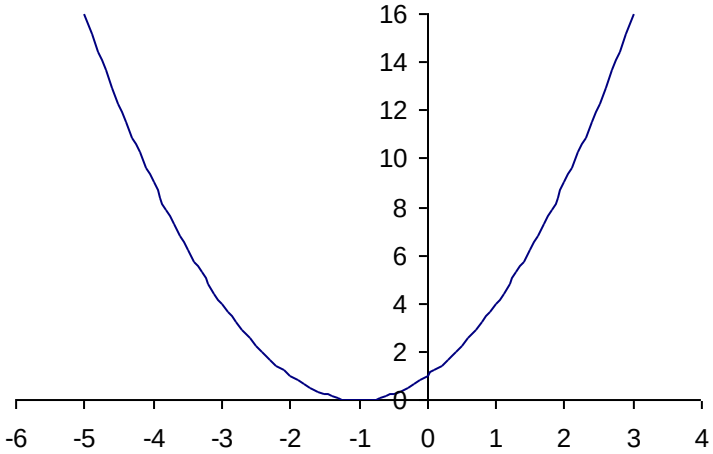
EXAMPLE

Sketch the graph of a function that has the properties described.

$f(-1) = 0$; $f'(x) < 0$ for $x < -1$; $f'(-1) = 0$ and $f'(x) > 0$ for $x > -1$.

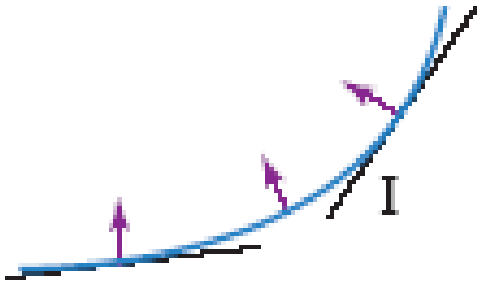
SOLUTION

The only specific point that the graph must pass through is $(-1, 0)$. Further, we know that to the left of this point, the graph must be decreasing ($f'(x) < 0$ for $x < -1$) and to the right of this point, the graph must be increasing ($f'(x) > 0$ for $x > -1$). Lastly, the graph must have zero slope at that given point ($f'(-1) = 0$).

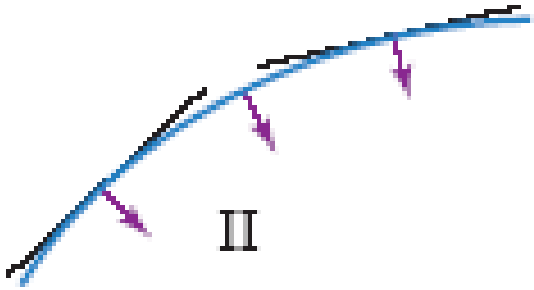


Second Derivative Rule

Second Derivative Rule If $f''(a) > 0$, then $f(x)$ is concave up at $x = a$. If $f''(a) < 0$, then $f(x)$ is concave down at $x = a$.

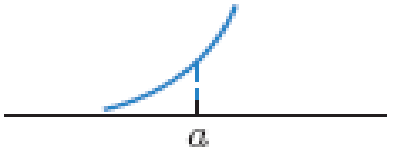
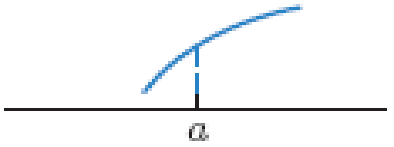
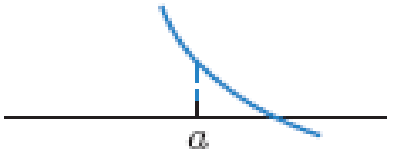
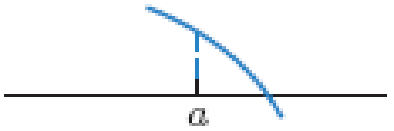


$f''(x) > 0$



$f''(x) < 0$

First & Second Derivative Scenarios

Conditions on the Derivatives	Description of $f(x)$ at $x = a$	Graph of $y = f(x)$ Near $x = a$
1. $f'(a)$ positive $f''(a)$ positive	$f(x)$ increasing $f(x)$ concave up	
2. $f'(a)$ positive $f''(a)$ negative	$f(x)$ increasing $f(x)$ concave down	
3. $f'(a)$ negative $f''(a)$ positive	$f(x)$ decreasing $f(x)$ concave up	
4. $f'(a)$ negative $f''(a)$ negative	$f(x)$ decreasing $f(x)$ concave down	

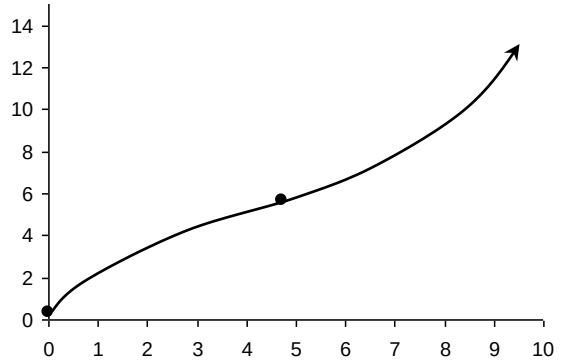
First & Second Derivative Rules

EXAMPLE

Sketch the graph of a function that has the properties described: $f(x)$ defined only for $x \geq 0$; (0, 0) and (5, 6) are on the graph; $f'(x) > 0$ for $x \geq 0$; $f''(x) < 0$ for $x < 5$, $f''(5) = 0$, $f''(x) > 0$ for $x > 5$.

SOLUTION

The only specific points that the graph must pass through are (0, 0) and (5, 6). Further, we know that to the left of (5, 6), the graph must be concave down ($f''(x) < 0$ for $x < 5$) and to the right of this point, the graph must be concave up ($f''(x) > 0$ for $x > 5$). Also, the graph will only be defined in the first and fourth quadrants ($x \geq 0$). Lastly, the graph must have positive slope everywhere that it is defined.



First Derivative Test

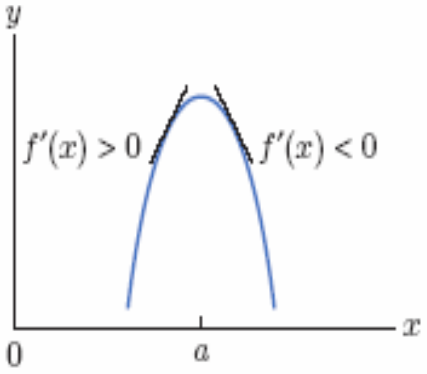
The First Derivative Test (for local extreme points)

Suppose that $f'(a) = 0$.

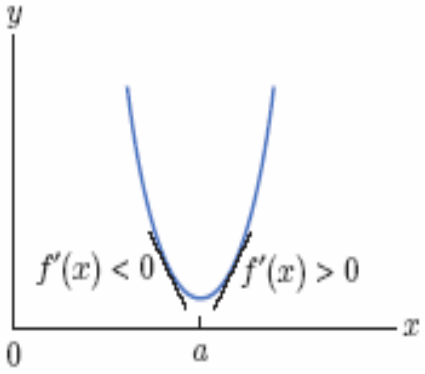
(a) If f' changes from positive to negative at $x = a$, then f has a local maximum at a (Fig. 1(a)).

(b) If f' changes from negative to positive at $x = a$, then f has a local minimum at a (Fig. 1(b)).

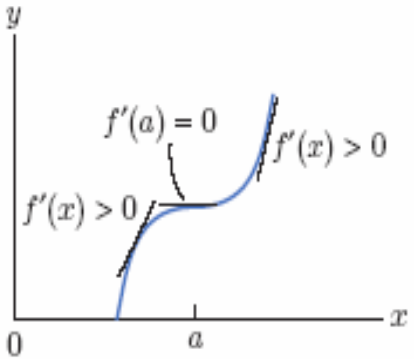
(c) If f' does not change sign at a (that is, f' is either positive on both sides of a (Fig. 1(c)), or negative on both sides of a (Fig. 1(d)), then f has no local extremum at a.



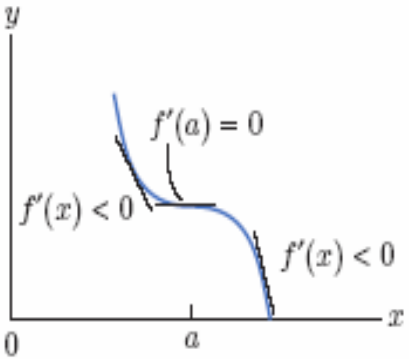
(a) Local maximum



(b) Local minimum



(c) No extreme point



(d) No extreme point

First Derivative Test

EXAMPLE

Find the local maximum and minimum of $f(x) = 2x^3 - 3x^2 - 12x + 5$.

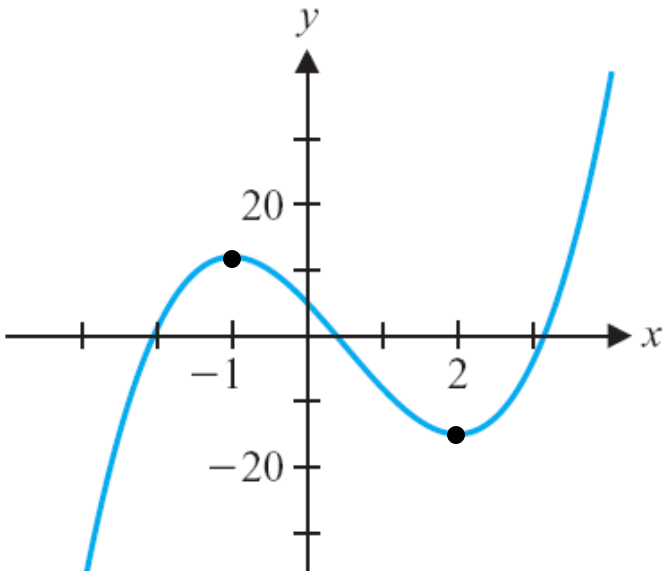
SOLUTION

$$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$$
$$= 6(x - 2)(x + 1)$$

Critical numbers: $x = -1, 2$

Interval	$(-\infty, -1)$	$(-1, 2)$	$(2, \infty)$
Sign of f'	+	−	+
Conclusion	↗	↘	↗

The local maximum is $f(-1) = 12$ and the local minimum is $f(2) = -15$

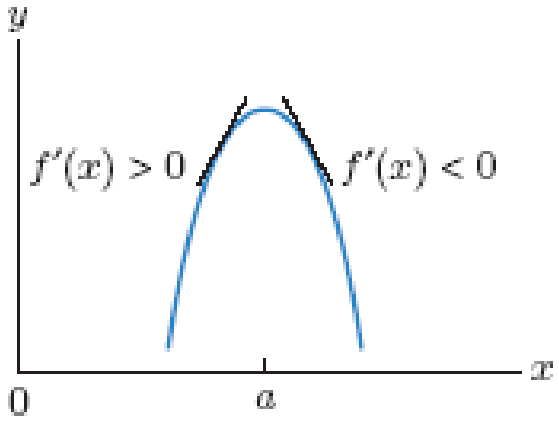


Second Derivative Test

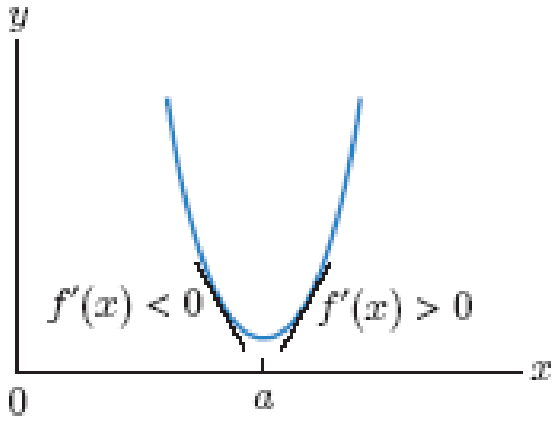
The Second Derivative Test (for local extreme points)

(a) If $f'(a) = 0$ and $f''(a) < 0$, then f has a local maximum at a (Fig. 1(a)).

(b) If $f'(a) = 0$ and $f''(a) > 0$, then f has a local minimum at a (Fig. 1(b)).



(a) Local maximum



(b) Local minimum

Second Derivative Test

EXAMPLE

Locate all possible relative extreme points on the graph of the function $f(x) = x^3 + 6x^2 + 9x$. Check the concavity at these points and use this information to sketch the graph of $f(x)$.

SOLUTION

We have

$$f(x) = x^3 + 6x^2 + 9x$$

$$f'(x) = 3x^2 + 12x + 9$$

$$f''(x) = 6x + 12.$$

The easiest way to find the critical values is to factor the expression for $f'(x)$:

$$3x^2 + 12x + 9 = (3x + 9)(x + 1).$$

Second Derivative Test

CONTINUED

From this factorization it is clear that $f'(x)$ will be zero if and only if $x = -3$ or $x = -1$. In other words, the graph will have horizontal tangent lines when $x = -3$ and $x = -1$, and no where else. To plot the points on the graph where $x = -3$ and $x = -1$, we substitute these values back into the original expression for $f(x)$. That is, we compute

$$f(-3) = (-3)^3 + 6(-3)^2 + 9(-3) = 0$$

$$f(-1) = (-1)^3 + 6(-1)^2 + 9(-1) = -4.$$

Therefore, the slope of $f(x)$ is 0 at the points $(-3, 0)$ and $(-1, -4)$.

Next, we check the sign of $f''(x)$ at $x = -3$ and at $x = -1$ and apply the second derivative test:

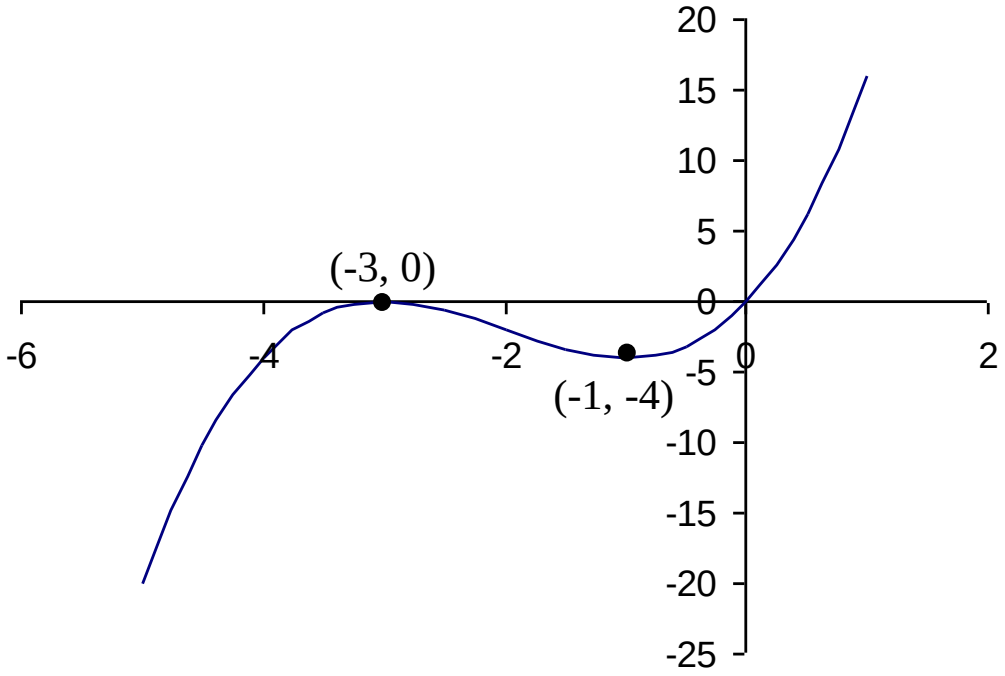
$$f''(-3) = 6(-3) + 12 = -6 < 0 \quad \text{(local maximum)}$$

$$f''(-1) = 6(-1) + 12 = 6 > 0 \quad \text{(local minimum).}$$

Second Derivative Test

CONTINUED

The following is a sketch of the function.



Test for Inflection Points

Look for possible inflection points by setting $f''(x) = 0$ and solving for x.

Second Derivative Test

EXAMPLE

Sketch the graph of $f(x) = x^3 - x + 2$.

SOLUTION

We have

$$f(x) = x^3 - x + 2$$

$$f'(x) = 3x^2 - 1$$

$$f''(x) = 6x.$$

We set $f'(x) = 0$ and solve for x.

$$3x^2 - 1 = 0$$

$$3x^2 = 1$$

$$x^2 = 1/3$$

$$x = \pm\frac{\sqrt{3}}{3} \qquad \text{(critical values)}$$

Second Derivative Test

CONTINUED

Substituting these values of x back into $f(x)$, we find that

$$f\left(\frac{\sqrt{3}}{3}\right)=\left(\frac{\sqrt{3}}{3}\right)^3-\left(\frac{\sqrt{3}}{3}\right)+2=\frac{18-2\sqrt{3}}{9}$$

$$f\left(-\frac{\sqrt{3}}{3}\right)=\left(-\frac{\sqrt{3}}{3}\right)^3-\left(-\frac{\sqrt{3}}{3}\right)+2=\frac{18+2\sqrt{3}}{9}.$$

We now compute

$$f''\left(\frac{\sqrt{3}}{3}\right)=6\left(\frac{\sqrt{3}}{3}\right)=2\sqrt{3}>0 \qquad \text{(local minimum)}$$

$$f''\left(-\frac{\sqrt{3}}{3}\right)=6\left(-\frac{\sqrt{3}}{3}\right)=-2\sqrt{3}<0 \qquad \text{(local maximum)}$$

Second Derivative Test

CONTINUED

Since the concavity reverses somewhere between $x = \frac{\sqrt{3}}{3}$ and $x = -\frac{\sqrt{3}}{3}$, there must be at least one inflection point. If we set $f''(x) = 0$, we find that

$$6x = 0$$
$$x = 0.$$

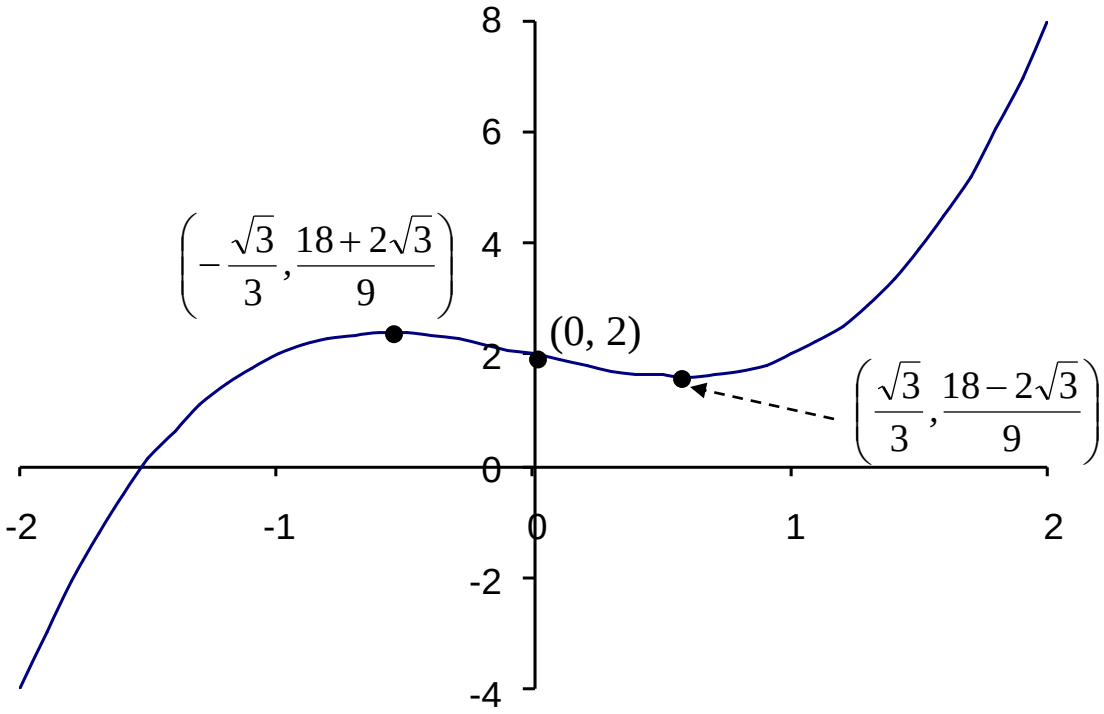
So the inflection point must occur at $x = 0$. In order to plot the inflection point, we compute

$$f(0) = (0)^3 - (0) + 2 = 2.$$

The final sketch of the graph is given below.

Second Derivative Test

CONTINUED



Second Derivative Test

EXAMPLE

Sketch the graph of $f(x) = x^3 - 6x^2 + 12x - 5$.

SOLUTION

We have

$$f(x) = x^3 - 6x^2 + 12x - 5$$
$$f'(x) = 3x^2 - 12x + 12$$
$$f''(x) = 6x - 12.$$

We set $f'(x) = 0$ and solve for x.

$$3x^2 - 12x + 12 = 0$$
$$(3x - 6)(x - 2) = 0$$
$$x = 2 \qquad \text{(critical value)}$$

Second Derivative Test

CONTINUED

Since $f''(2) = 0$, we know nothing about the graph at $x = 2$. However, the test for inflection points suggests that we have an inflection point at $x = 2$. First, let's verify that we indeed have an inflection point at $x = 2$. If this proves to be not the case, we would use a similar method (using the first derivative) to see if we have a relative extremum at $x = 2$.

Notice, $x = 2$ was the only candidate for generating a relative extremum. Therefore, there are no relative extrema. We will now find the y-coordinate for the inflection point.

$$f(2) = (2)^3 - 6(2)^2 + 12(2) - 5 = 3$$

So, the only inflection point is at (2, 3).

Second Derivative Test

CONTINUED

Now we will look for intercepts. Let's first look for a *y*-intercept by evaluating $f(0)$.

$$f(0) = (0)^3 - 6(0)^2 + 12(0) - 5 = -5$$

So, we have a *y*-intercept at (0, –5). To find any *x*-intercepts, we replace $f(x)$ with 0.

$$0 = x^3 - 6x^2 + 12x - 5$$

Since this equation does not factor, and the quadratic formula cannot help us either, we attempt to use the Rational Roots Theorem from algebra. In doing so we find that there are no rational roots (*x*-intercepts). So, if there is an *x*-intercept, it will be an irrational number. Below, we show *some* of the work employed in estimating the *x*-intercept.

Second Derivative Test

CONTINUED

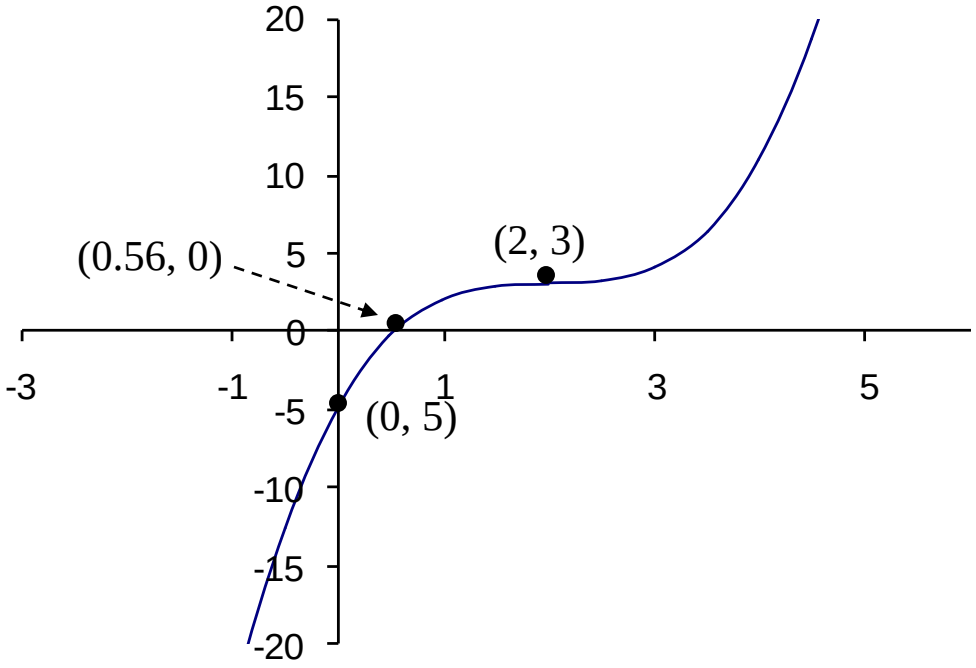
x	$f(x)$
0.54	−0.11
0.55	−0.05
0.56	0.01
0.57	0.08

Notice that the y-values corresponding to $x = 0.54$ and $x = 0.55$ are below the x-axis and the y-values corresponding to $x = 0.56$ and $x = 0.57$ are above the x-axis. Therefore, in between $x = 0.55$ and $x = 0.56$, there must be an x-intercept. For the sake of brevity, we'll just take $x = 0.56$ for our x-intercept since, out of the four x-values above, it has the y-value closest to zero. Therefore, the point of our x-intercept is (0.56, 0).

Now we will sketch a graph of the function.

Second Derivative Test

CONTINUED



Graphs on closed intervals

EXAMPLE

Let $f(x) = x^3 - 3x^2 - 9x + 1, -2 \le x \le 6$.

a) Find the intervals on which the function f is increasing or decreasing and find the local maximum and minimum, if any.

b) Find the intervals on which the graph of f is concave up or concave down and find the points of inflection, if any.

c) What is the absolute maximum? Minimum?

d) Sketch the graph of f.

SOLUTION

$$f'(x) = 3x^2 - 6x - 9 = 0 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x+1)(x-3) = 0$$

Interval	$(-2, -1)$	$(-1, 3)$	$(3, 6)$
Sign of f'	+	–	+
Conclusion	↗	↘	↗

Local maximum: $(-1, 6)$ & $(6, 55)$

Local minimum: $(-2, -1)$ & $(3, -26)$

Absolute maximum: $(6, 55)$

Absolute minimum: $(3, -26)$

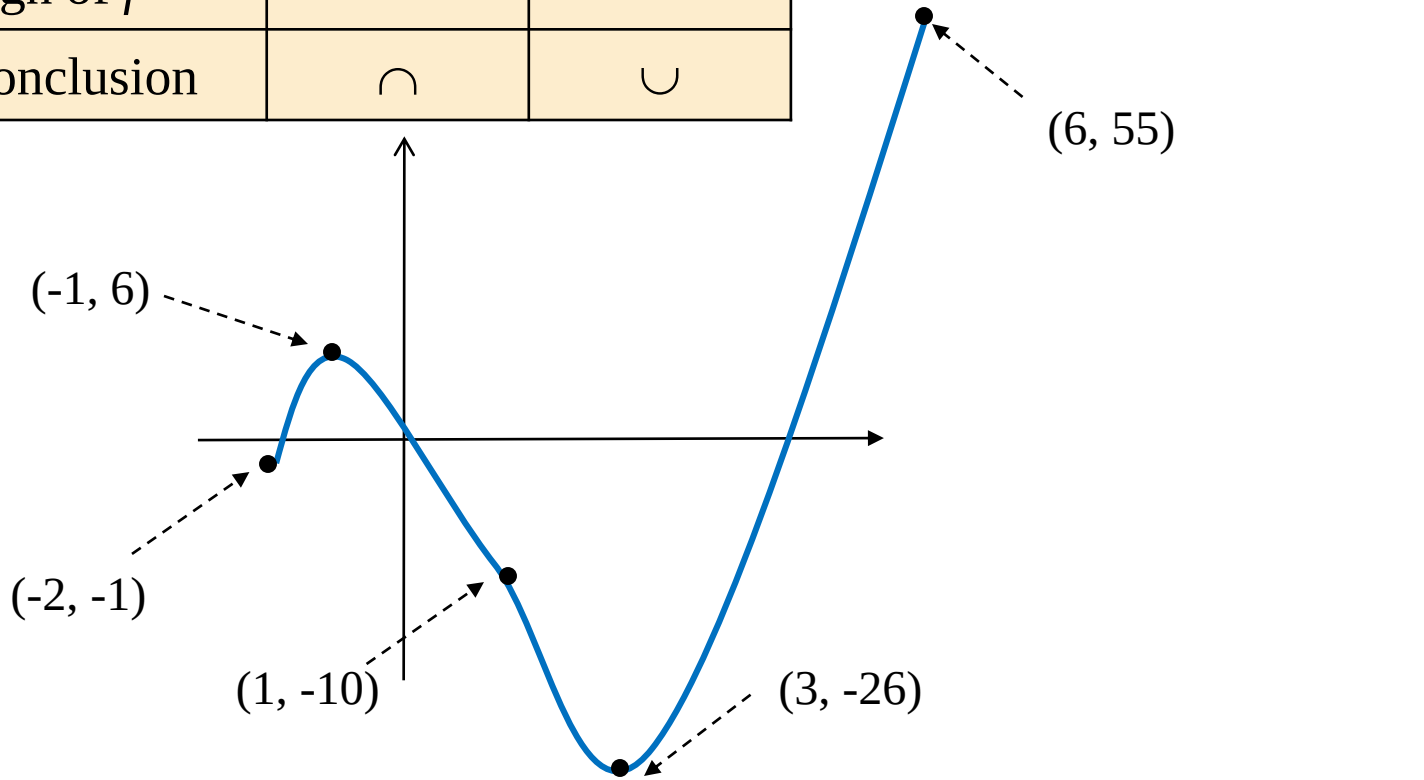
Graphs on closed intervals

CONTINUED

$$f''(x) = 6x^2 - 6 = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$$

Interval	(–2, 1)	(1, 6)
Sign of f''	–	+
Conclusion	∩	∪

Inflection point: (1, –10)



Optimization Problems

EXAMPLE

Find two positive numbers x and y that maximize $Q = x^2y$ if $x + y = 6$.

SOLUTION

Solving $x + y = 6$ for y gives $y = 6 - x$. Substituting into $Q = x^2y$ yields

$$Q(x) = x^2(6 - x) = 6x^2 - x^3$$

$$\frac{dQ}{dx} = 0 \Rightarrow 12x - 3x^2 = 0 \Rightarrow 3x(4 - x) = 0 \Rightarrow x = 0 \text{ or } x = 4$$

$$\frac{d^2Q}{dx^2} = 12 - 6x, \left.\frac{d^2Q}{dx^2}\right|_{x=0} = 12 > 0, \left.\frac{d^2Q}{dx^2}\right|_{x=4} = -12 < 0$$

The maximum value of Q occurs at $x = 4$ and $y = 2$.

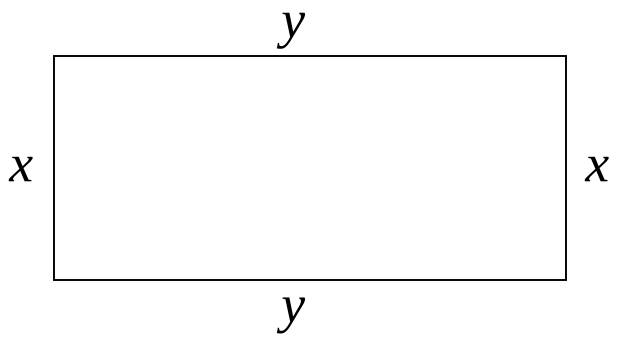
Maximizing Area

EXAMPLE

Find the dimensions of the rectangular garden of greatest area that can be fenced off (all four sides) with 300 meters of fencing.

SOLUTION

Let's start with what we know. The garden is to be in the shape of a rectangle. The perimeter of it is to be 300 meters. Let's make a picture of the garden, labeling the sides.



Since we know the perimeter is 300 meters, we can now construct an equation based on the variables contained within the picture.

$$x + x + y + y = 2x + 2y = 300 \qquad \text{(Constraint Equation)}$$

Maximizing Area

CONTINUED

Now, the quantity we wish to maximize is area. Therefore, we will need an equation that contains a variable representing area. This is shown below.

$$A = xy \qquad \text{(Objective Equation)}$$

Now we will rewrite the objective equation in terms of A (the variable we wish to optimize) and either x or y. We will do this, using the constraint equation. Since it doesn't make a difference which one we select, we will select x.

$2x + 2y = 300$	This is the constraint equation.
$2y = 300 - 2x$	Subtract.
$y = 150 - x$	Divide.

Now we substitute $150 - x$ for y in the objective equation so that the objective equation will have *only one independent variable*.

Maximizing Area

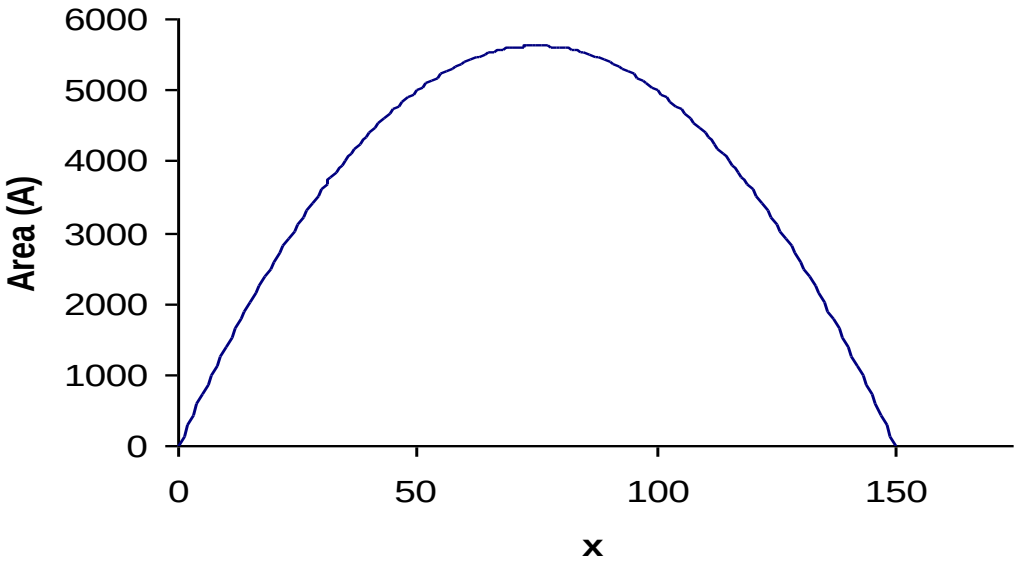
CONTINUED

$A = xy$ This is the objective equation.

$A = x(150 - x)$ Replace y with $150 - x$.

$A = 150x - x^2$ Distribute.

Now we will graph the resultant function, $A = 150x - x^2$.



Maximizing Area

CONTINUED

Since the graph of the function is obviously a parabola, then the maximum value of A (along the vertical axis) would be found at the only value of x for which the first derivative is equal to zero.

$A = 150x - x^2$	This is the area function.
$A' = 150 - 2x$	Differentiate.
$150 - 2x = 0$	Set the derivative equal to 0.
$x = 75$	Solve for x.

Therefore, the slope of the function equals zero when $x = 75$. Therefore, that is the x-value for where the function is maximized. Now we can use the constraint equation to determine y.

$$2x + 2y = 300 \qquad 2(75) + 2y = 300 \qquad y = 75$$

So, the dimensions of the garden will be 75 m x 75 m.

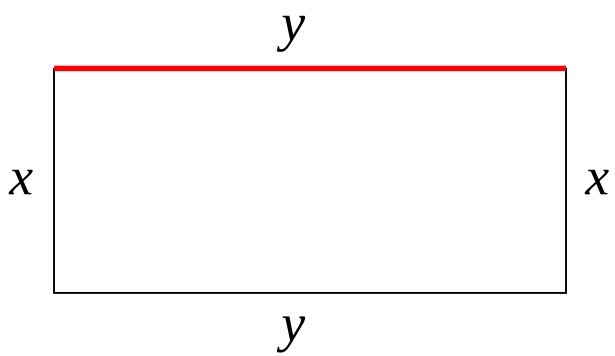
Minimizing Cost

EXAMPLE

A rectangular garden of area 75 square feet is to be surrounded on three sides by a brick wall costing $10 per foot and on one side by a fence costing $5 per foot. Find the dimensions of the garden such that the cost of materials is minimized.

SOLUTION

Below is a picture of the garden. The red side represents the side that is fenced.



The quantity that we will be minimizing is 'cost'. Therefore, our objective equation will contain a variable representing cost, C.

Minimizing Cost

CONTINUED

$$C = (2x + y)(10) + y(5)$$

$$C = 20x + 10y + 5y$$

$$C = 20x + 15y \qquad \text{(Objective Equation)}$$

Now we will determine the constraint equation. The only piece of information we have not yet used in some way is that the area is 75 square feet. Using this, we create a constraint equation as follows.

$$75 = xy \qquad \text{(Constraint Equation)}$$

Now we rewrite the constraint equation, isolating one of the variables therein.

$$75 = xy$$

$$75/y = x$$

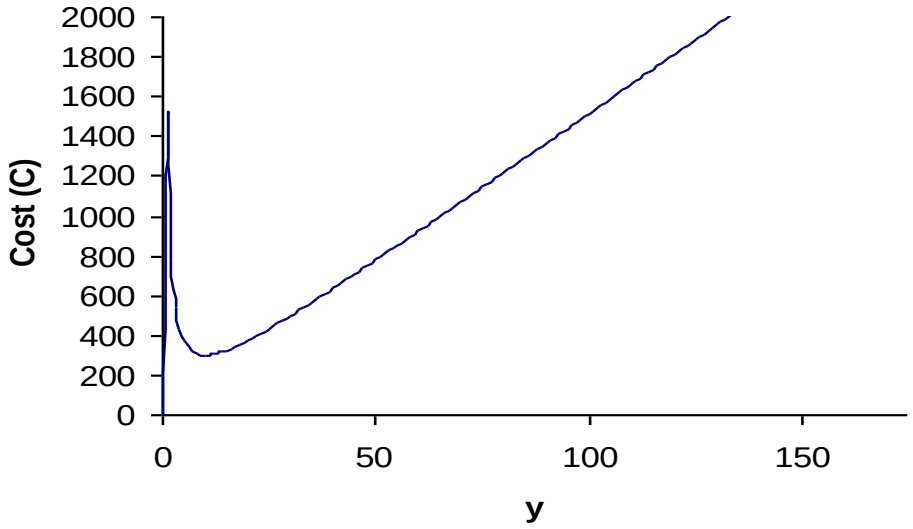
Minimizing Cost

CONTINUED

Now we rewrite the objective equation using the substitution we just acquired from the constraint equation.

$C = 20x + 15y$	This is the objective equation.
$C = 20(75/y) + 15y$	Replace x with $75/y$.
$C = 1500/y + 15y$	Simplify.

Now we use this equation to sketch a graph of the function.



Minimizing Cost

CONTINUED

It appears from the graph that there is exactly one relative extremum, a relative minimum around $x = 10$ or $x = 15$. To know exactly where this relative minimum is, we need to set the first derivative equal to zero and solve (since at this point, the function will have a slope of zero).

$C = 1500/y + 15y$	This is the given equation.
$C' = -1500/y^2 + 15$	Differentiate.
$-1500/y^2 + 15 = 0$	Set the function equal to 0.
$15 = 1500/y^2$	Add.
$15y^2 = 1500$	Multiply.
$y^2 = 100$	Divide.
$y = 10$	Take the positive square root of both sides (since $y > 0$).

Minimizing Cost

CONTINUED

Therefore, we know that cost will be minimized when $y = 10$. Now we will use the constraint equation to determine the corresponding value for x.

$75 = xy$	This is the constraint equation.
$75 = x(10)$	Replace y with 10.
$7.5 = x$	Solve for x.

So the dimensions that will minimize cost, are $x = 7.5$ ft and $y = 10$ ft.

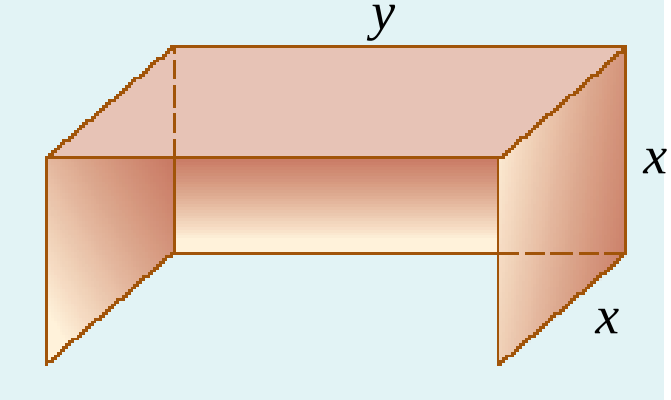
Minimizing Surface Area

EXAMPLE

(*Volume*) A canvas wind shelter for the beach has a back, two square sides, and a top. Find the dimensions for which the volume will be 250 cubic feet and that requires the least possible amount of canvas.

SOLUTION

Below is a picture of the wind shelter.



The quantity that we will be minimizing is 'surface area'. Therefore, our objective equation will contain a variable representing surface area, A.

Minimizing Surface Area

CONTINUED

$A = xx + xx + xy + xy$ Sum of the areas of the sides

$A = 2x^2 + 2xy$ (Objective Equation)

Now we will determine the constraint equation. The only piece of information we have not yet used in some way is that the volume is 250 ft^3. Using this, we create a constraint equation as follows.

$250 = x^2y$ (Constraint Equation)

Now we rewrite the constraint equation, isolating one of the variables therein.

$$250 = x^2y$$

$$250/x^2 = y$$

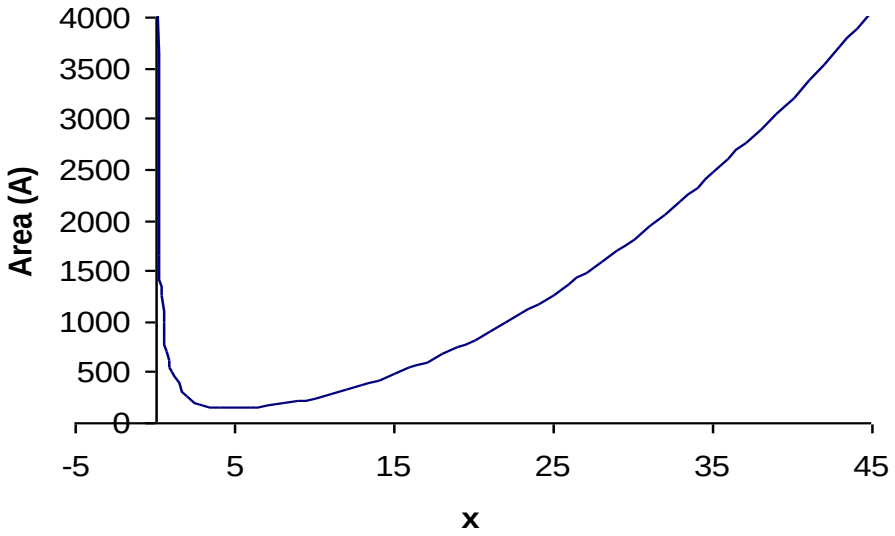
Minimizing Surface Area

CONTINUED

Now we rewrite the objective equation using the substitution we just acquired from the constraint equation.

$A = 2x^2 + 2xy$	This is the objective equation.
$A = 2x^2 + 2x(250/x^2)$	Replace y with $250/x^2$.
$A = 2x^2 + 500/x$	Simplify.

Now we use this equation to sketch a graph of the function.



Minimizing Surface Area

CONTINUED

It appears from the graph that there is exactly one relative extremum, a relative minimum around $x = 5$. To know exactly where this relative minimum is, we need to set the first derivative equal to zero and solve (since at this point, the function will have a slope of zero).

$A = 2x^2 + 500/x$	This is the given equation.
$A' = 4x - 500/x^2$	Differentiate.
$4x - 500/x^2 = 0$	Set the function equal to 0.
$4x = 500/x^2$	Add.
$4x^3 = 500$	Multiply.
$x^3 = 125$	Divide.
$x = 5$	Take the cube root of both sides.

Minimizing Surface Area

CONTINUED

Therefore, we know that surface area will be minimized when $x = 5$. Now we will use the constraint equation to determine the corresponding value for y.

$250 = x^2y$	This is the constraint equation.
$250 = (5)^2y$	Replace x with 5.
$10 = y$	Solve for y.

So the dimensions that will minimize surface area, are $x = 5$ ft and $y = 10$ ft.